JEE (ADVANCED) 2021 PAPER-2

[PAPER WITH SOLUTION]

HELD ON SUNDAY 03RD OCTOBER2021

MATHEMATICS

SECTION1 (Maximum Marks :24)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is(are) correct option(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : **+4** If only (all) the correct option(s) is(are) chosen.

Partial Marks : **+3** If all the four options are correct but ONLY three options are chosen.

Partial Marks : **+2** If three or more options are correct but ONLY two options are chosen and both ofwhich are correct.

Partial Marks : **+1** If two or more options are correct but ONLY one option is chosen and it is a correct option.

Zero Marks : **0** If none of the options is chosen (i.e. the question is unanswered).

Negative Marks :**–2** In all other cases.

- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then

choosing ONLY (A), (B) and (D) will get +4 marks;

choosing ONLY (A) and (B) will get +2 marks;

choosing ONLY (A) and (D) will get +2marks;

choosing ONLY (B) and (D) will get +2 marks;

choosing ONLY (A) will get +1 mark;

choosing ONLY (B) will get +1 mark;

choosing ONLY (D) will get +1 mark;

choosing no option(s) (i.e. the question is unanswered) will get 0 marks and

choosing any other option(s) will get −2 marks.

[:Q.1] Let $S_1 = \{(i,j,k) : i,j,k \in \{1,2,....,10\}\}$,

$S_2 = \{(i,j) : 1 \le i < j+2 \le 10, i,j \in \{1,2,.....,10\}\}$,

$S_3 = \{(i,j,k,l) : 1 \le i < j < k < l, i,j,k,l \in \{1,2,.....,10\}\}$,

and $S_4 = \{(i,j,k,l) : i,j,k \text{ and } l \text{ are distinct elements in } \{1,2,......,10\}\}$.

If the total number of elements in the set S_r is n_r, r = 1,2,3,4, then which of the following statements is (are) TRUE ?

[:A] $n_1 = 1000$

[:B] $n_2 = 44$

[:C] $n_3 = 220$

[:D] $\frac{n_4}{12} = 420$

[:ANS] A,B,D

[:SOLN] Number of elements in $S_1 = n_1 = 10 \times 10 \times 10 = 1000$

Number of elements in $S_2 = n_2 = 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 = 44$

Number of elements in $S_3 = n_3 = {}^{10}C_4 = 210$

Number of elements in $S_4 = n_4 = {}^{10}C_4 \cdot \underline{|4} = 210 \times \underline{|4} = 5040$

[:Q.2] Consider a triangle PQR having sides of lengths p, and r opposite to the angles P, and R, respectively. Then which of the following statements is (are) TRUE ?

[:A] $\cos P \ge 1 - \frac{p^2}{2qr}$

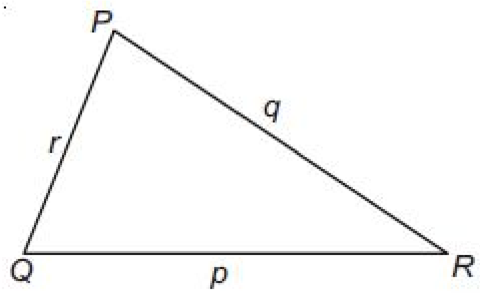
[:B] $\cos R \ge \left(\frac{q-r}{p+q}\right)\cos P + \left(\frac{p-r}{p+q}\right)\cos Q$

[:C] $\frac{q+r}{p} < 2\frac{\sqrt{\sin Q \sin R}}{\sin P}$

[:D] If p < q and p < r, then $\frac{q+r}{p} < 2\frac{\sqrt{\sin Q \sin R}}{\sin P}$

[:ANS] A,B

[:SOLN]



(A) $\cos P = \frac{q^2+r^2-p^2}{2qr}$ here $\frac{q^2+r^2}{2} \geq \sqrt{q^2.r^2}$ (using $A.M \geq G.M$)

$\Rightarrow \quad q^2+r^2 \geq 2qr$

Hence, $\cos P \geq \frac{2qr-p^2}{2qr}$

$\cos P \geq 1-\frac{p^2}{2qr}$

(B) $\frac{(q-r)\cos P+(p-r)\cos Q}{p+q} = \frac{(q\cos P+p\cos Q)-r(\cos P+\cos Q)}{p+q}$

$= \frac{r(1-\cos P-\cos Q)}{p+q} = \frac{r(q-p\cos R)-(p-q\cos R)}{p+q} = \frac{(r-p-q)+(p+q)\cos R}{p+q}$

$= \cos R + \frac{r-q-p}{p+q} \leq \cos R \Rightarrow \cos R + \frac{r-(p+q)}{p+q} \leq \cos R$ (Since (p + q > r))

(C) $\frac{q+r}{p} = \frac{\sin Q+\sin R}{\sin P} \geq \frac{2\sqrt{\sin Q.\sin R}}{\sin P}$ (using $A.M \geq G.M$)

(D) If p < q and q < r

So, p, is the smallest side, therefore one of Q or R can be obtuse

So, one of cosQ or cosR can be negative

Therefore $\cos Q > \frac{p}{r}$ and $\cos R > \frac{p}{q}$ cannot satisfy always.

[:Q.3] Let $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right] \to \mathbb{R}$ be a continuous function such that $f(0)=1$ and $\int_0^{\frac{\pi}{3}} f(t)dt = 0$

Then which of the following statements is (are) TRUE ?

[:A] The equation f(x) – 3 cos 3x = 0 has at least one solution in $\left(0,\frac{\pi}{3}\right)$

[:B] The equation f(x) – 3 cos 3x = $-\frac{6}{\pi}$ has at least one solution in $\left(0,\frac{\pi}{3}\right)$

[:C] $\lim_{x\to 0}\frac{x\int_0^x f(t)\,dt}{1-e^{x^2}}=-1$

[:D] $\lim_{x\to 0}\frac{\sin x\int_0^x f(t)\,dt}{x^2}=-1$

[:ANS] A,B,C

[:SOLN] $f(0)=1\ \&\ \int_0^{\frac{\pi}{3}} f(t)\,dt=0$

Consider a function $g(x)=\int_0^x f(t)\,dt-\sin 3x$

Since g(x) is continous and differentiable function

Also, g(0) = 0 & g(π/3) = 0

According to Rolle's theorem $g'(x)=0$ has at least one solution in $\left(0,\frac{\pi}{3}\right)$

Hence $f(x)-3\cos(3x)=0$ for some $x\in\left(0,\frac{\pi}{3}\right)$

(B) Consider a function $\phi(x)=\int_0^x f(t)\,dt+\cos 3x+\frac{6}{\pi}x$ since $\phi(x)$ is continuous and differentiable function and $\phi(0)=1\ \&\ \phi\left(\frac{\pi}{3}\right)=1$

Using Rolle's Theorem $\phi'(x)=0$ for at least one $x\in\left(0,\frac{\pi}{3}\right)$

(C) $\lim_{x\to 0}\frac{x\int_0^x f(t)\,dt}{1-e^{x^2}},\left(\frac{0}{0}\text{form}\right)$

By L' Hopital rule $\lim_{x\to 0}\frac{xf(x)+\int_0^x f(t)\,dt}{-2xe^{x^2}},\left(\frac{0}{0}\text{form}\right)$

$$=\lim_{x\to 0}\frac{xf'(x)+f(x)+f(x)}{-4x^2e^{x^2}-2e^{x^2}}=\frac{0+2f(0)}{-0-2}=-1$$

(D) $\lim_{x\to 0}\frac{\sin x\int_0^x f(t)\,dt}{x^2},\left(\frac{0}{0}\text{form}\right)$

$$= \lim_{x\to 0} \frac{\sin x . f(x) + \cos x \int_0^x f(t)\,dt}{2x}$$

$$= \lim_{x\to 0} \frac{\left(\cos x . f(x) + \sin x . f'(x) + \cos x . f(x) - \sin x . \int_0^x f(t)\,dt\right)}{2}$$

$$= \frac{1+0+1-0}{2} = 1.$$

[:Q.4] For any real numbers α and β, let $y_{\alpha,\beta}(x)$, $x \in \mathbb{R}$, be the solution of the differential equation $\frac{dy}{dx} + \alpha y = xe^{\beta x}, y(1) = 1$. Let $S = \{y_{\alpha,\beta}(x) : \alpha, \beta \in \mathbb{R}\}$. Then which of the following functions belong(s) to the set S ?

[:A] $f(x) = \frac{x^2}{2}e^{-x} + \left(e - \frac{1}{2}\right)e^{-x}$

[:B] $f(x) = -\frac{x^2}{2}e^{-x} + \left(e + \frac{1}{2}\right)e^{-x}$

[:C] $f(x) = \frac{e^x}{2}\left(x - \frac{1}{2}\right) + \left(e - \frac{e^2}{4}\right)e^{-x}$

[:D] $f(x) = \frac{e^x}{2}\left(\frac{1}{2} - x\right) + \left(e + \frac{e^2}{4}\right)e^{-x}$

[:ANS] **A,C**

[:SOLN] $\frac{dy}{dx} + \alpha y = xe^{\beta x}$

Integrating factor $(\text{I.F.}) = e^{\int \alpha dx} = e^{\alpha x}$

Hence, the solution is $y.e^{\alpha x} = \int xe^{\beta x}.e^{\alpha x}dx$

$ye^{\alpha x} = \int xe^{(\alpha+\beta)x}dx$

If $\alpha + \beta \neq 0$

$ye^{\alpha x} = x\frac{e^{(\alpha+\beta)x}}{(\alpha+\beta)} - \frac{e^{(\alpha+\beta)x}}{(\alpha+\beta)^2} + C$ (Here $(\alpha+\beta) \neq 0$)

$$y=\frac{xe^{\beta x}}{(\alpha+\beta)}-\frac{e^{\beta x}}{(\alpha+\beta)^2}+Ce^{-\alpha x}$$

$$y=\frac{e^{\beta x}}{(\alpha+\beta)}\left(x-\frac{1}{\alpha+\beta}\right)+Ce^{-\alpha x} \quad \text{.... (i)}$$

Put $\alpha=\beta=1$ in (i) we get

$$y=\frac{e^x}{2}\left(x-\frac{1}{2}\right)+Ce^{-x}$$

Since, $y(1)=1$

So, $1=\frac{e}{2}\times\frac{1}{2}+\frac{C}{e}\Rightarrow C=e-\frac{e^2}{4}$

So, $y=\frac{e^x}{2}\left(x-\frac{1}{2}\right)+\left(e-\frac{e^2}{4}\right)e^{-x}$

If $\alpha+\beta=0$ & $\alpha=1$

$$\frac{dy}{dx}+y=xe^{-x}$$

Integrating Factor $=e^x$

$$ye^x=\int xdx$$

$$ye^x=\frac{x^2}{2}+C$$

$$y=\frac{x^2}{2}e^{-x}+Ce^{-x}$$

Since, $y(1)=1$

$$1=\frac{1}{2e}+\frac{C}{e}\Rightarrow C=e-\frac{1}{2}$$

$$y=\frac{x^2}{2}e^{-x}+\left(e-\frac{1}{2}\right)e^{-x}$$

[:Q.5] Let O be the origin and $\overrightarrow{OA}=2\hat{i}+2\hat{j}+\hat{k}$, $\overrightarrow{OB}=\hat{i}-2\hat{j}+2\hat{k}$ and $\overrightarrow{OC}=\frac{1}{2}\left(\overrightarrow{OB}-\lambda\overrightarrow{OA}\right)$ for some $\lambda>0$.

If $\left|\overrightarrow{OB}\times\overrightarrow{OC}\right|=\frac{9}{2}$, , then which of the following statements is (are) TRUE ?

[:A] Projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$ is $-\frac{3}{2}$

[:B] Area of the triangle OAB is $\frac{9}{2}$

[:C] Area of the triangle ABC is $\frac{9}{2}$

[:D] The acute angle between the diagonals of the parallelogram with adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is $\frac{\pi}{3}$

[:ANS] A,B,C

[:SOLN] $\overrightarrow{OA} = 2\hat{i} + 2\hat{j} + \hat{k}$

$\overrightarrow{OB} = \hat{i} - 2\hat{j} + 2\hat{k}$

$\overrightarrow{OC} = \frac{1}{2}\left(\overrightarrow{OB} - \lambda\overrightarrow{OA}\right)$

$\overrightarrow{OB} \times \overrightarrow{OC} = \overrightarrow{OB} \times \frac{1}{2}\left(\overrightarrow{OB} - \lambda\overrightarrow{OA}\right) = -\frac{\lambda}{2}\overrightarrow{OB} \times \overrightarrow{OA} = \frac{\lambda}{2}\left(\overrightarrow{OA} \times \overrightarrow{OB}\right)$

Now, $\overrightarrow{OA} \times \overrightarrow{OB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{vmatrix} = 6\hat{i} - 3\hat{j} - 6\hat{k}$

So, $\overrightarrow{OB} \times \overrightarrow{OC} = \frac{3\lambda}{2}\left(2\hat{i} - \hat{j} - 2\hat{k}\right)$

$\left|\overrightarrow{OB} \times \overrightarrow{OC}\right| = \left|\frac{9\lambda}{2}\right| = \frac{9}{2}$

So, $\lambda = 1 \quad (\because \lambda > 0)$

$\overrightarrow{OC} = \frac{1}{2}\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$

$\overrightarrow{OC} = \frac{1}{2}\left(-\hat{i} - 4\hat{j} + \hat{k}\right)$

(A) Projection of $\overrightarrow{OC}$ on $\overrightarrow{OA} = \frac{\overrightarrow{OC}.\overrightarrow{OA}}{\left|\overrightarrow{OA}\right|} = \frac{\frac{1}{2}(-2-8+1)}{3} = -\frac{3}{2}$

(B) Area of the triangle $OAB = \frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right| = \frac{9}{2}$

(C) Area of the triangle ABC is $=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \frac{1}{2}\left\|\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & -4 & 1\\ -\frac{5}{2} & -4 & -\frac{1}{2}\end{vmatrix}\right\| = \frac{1}{2}\left|6\hat{i}-3\hat{j}-6\hat{k}\right| = \frac{9}{2}$

(D) Acute angle between the diagonals of the parallelogram with adjacent sides $\overrightarrow{OA}$ & $\overrightarrow{OC}$ is θ, hence $\cos\theta = \frac{\left(\overrightarrow{OA}+\overrightarrow{OC}\right).\left(\overrightarrow{OA}-\overrightarrow{OC}\right)}{\left|\overrightarrow{OA}+\overrightarrow{OC}\right|\left|\overrightarrow{OA}-\overrightarrow{OC}\right|}$

$$\cos\theta = \frac{\left(\frac{3}{2}\hat{i}+\frac{3}{2}\hat{k}\right).\left(\frac{5}{2}\hat{i}+4\hat{j}+\frac{1}{2}\hat{k}\right)}{\frac{3}{2}\sqrt{2}\times\sqrt{\frac{90}{4}}} = \frac{18}{3\sqrt{2}\sqrt{90}}$$

hence $\theta \neq \frac{\pi}{3}$.

[:Q.6] Let E denote the parabola $y^2=8x$. Let $P=(-2,4)$, and let Q and Q' be two distinct points on E such that the lines PQ and PQ' are tangents to E. Let F be the focus of E. Then which of the following statements is (are) TRUE ?

[:A] The triangle PFQ is a right-angled triangle

[:B] The triangle QPQ' is a right-angled triangle

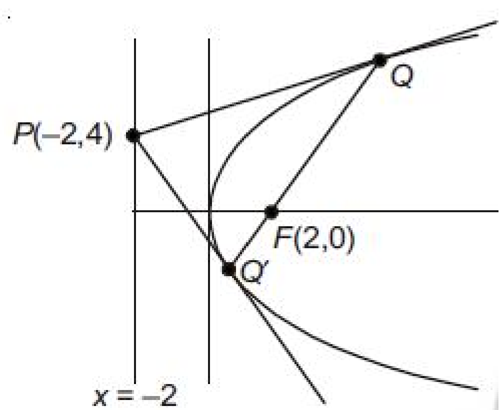
[:C] The distance between P and F is $5\sqrt{2}$

[:D] F lies on the line joining Q and Q'

[:ANS] **A,B,D**

[:SOLN] $E: y^2 = 8x$

$P:(-2,4)$



Since point P (–2, 4) lies on directrix (x = – 2) of parabola $y^2 = 8x$

So, $\angle QPQ' = \frac{\pi}{2}$ and chord QQ' is a focal chord and segment PQ subtends right angle at the focus.

Slope of $QQ' = \frac{2}{t_1 + t_2} = 1$

Slope of PF = – 1

$PF = 4\sqrt{2}.$

SECTION–2 (Maximum Marks : 12)

- This section contains **THREE (03)** question stems.
- There are **TWO (02)** questions corresponding to each question stem.
- The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme: Full Marks : +2 If ONLY the correct numerical value is entered at the designated place; Zero Marks : 0 In all other cases.

Question Stem for Question Nos. 7 and 8

Consider the region $R = \{(x, y) \in R \times R : x \geq 0 \text{ and } y^2 \leq 4 - x\}$. Let F be the family of all circles that are contained in R and have centers on the x-axis. Let C be the circle that has largest radius among the circles in F. Let (α, β) be a point where the circle C meets the curve $y^2 = 4 - x$.

[:Q.7] The radius of the circle C is _______________.

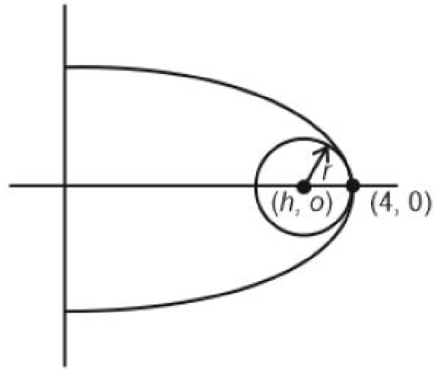
[:ANS] **00.50**

[:Q.8] The value of α is __________.

[:ANS] **04.00**

Sol. For comprehension 7 & 8

[:SOLN]



Parabola : $y^2 = 4 - x$ ...(i)

Circle : $(x - h)^2 + y^2 = r^2$ ...(ii)

Solving (i) and (ii), we get

$(x - h)^2 + 4 - x = r^2$

$\Rightarrow \quad x^2 - (2h + 1)x + (h^2 - r^2 + 4) = 0$ ...(iii)

At the common point of contact, (iii) will have equal roots.

$\therefore \quad (2h + 1)^2 = 4(h^2 - r^2 + 4)$

$\Rightarrow \quad 4r^2 = 15 - 4h$ ...(iv)

Let the common point of contact be $P(4 - t^2, t)$ which is same as (α, β).

Equation of tangent to parabola at P is $x + 2ty - (t^2 + 4) = 0$ ...(v)

and equation of tangent to circle is $4(4 - t^2 - h)x + 4ty + (4h^2 + (4t^2 - 12)h - 15) = 0$ ...(vi)

Since (v) and (vi) represent the same equation, on comparing the coefficients, we get

$h = 0$ and $r = \dfrac{\sqrt{15}}{2}$ (If $t \neq 0$) which is rejected because part of circle lies outside R.

$\therefore \quad t = 0$

$\Rightarrow \quad P \equiv (4,0) \equiv (\alpha, \beta)$

$\Rightarrow \quad \boxed{\alpha = 4}$

So equation of common tangent is $x = 4$.

If $x = 4$ is tangent to circle, then radius $r = 4 - h$ ....(vii)

Solving (iv) and (vii), we get

$4r^2 = 15 - 4(4 - r)$

$\Rightarrow 4r^2 - 4r + 1 = 0$

$\Rightarrow \boxed{r = \frac{1}{2}}$

Question Stem for Question Nos. 9 and 10

Let $f_1 : (0,\infty) \to \mathbb{R}$ and $f_2 : (0,\infty) \to \mathbb{R}$ be defined by $f_1(x) = \int_0^x \prod_{j=1}^{21} (t-j)^j \, dt, x > 0$

And $f_2(x) = 98(x-1)^{50} - 600(x-1)^{49} + 2450, x > 0,$ where, for any positive integer n and real numbers $a_1, a_2, \ldots a_n$, $\prod_{j=1}^{n} a_i$ denotes the product of $a_1, a_2, \ldots a_n$. Let m_i and n_i, respectively, denote the number of points of local minima and the number of points of local maxima of function f_i, *i* = 1, 2, in the interval $(0,\infty)$.

[:Q.9] The value of $2m_1 + 3n_1 + m_1n_1$ is ____________.

[:ANS] 57.00

$2m_1 + 3n_1 + m_1n_1$

= 2 × 6 + 3 × 5 + 6 × 5

= 57

[:Q.10] The value of $6m_2 + 4n_2 + 8m_2n_2$ is _________.

[:ANS] 06.00

$6m_2 + 4n_2 + 8m_2n_2$

= 6 × 1 + 4 × 0 + 8 × 1 × 0 = 6

Solution for Q. no. 9 & 10

[:SOLN]

$$f_1'(x) = \prod_{j=1}^{21} (x-j)^j$$

$$f_1'(x) = (x-1)(x-2)^2(x-3)^3 \ldots (x-20)^{20}(x-21)^{21}$$

Checking the sign scheme of $f_1'(x)$ at

x = 1, 2, 3...21, we get

$f_1(x)$ has local minima at x = 1, 5, 9, 13, 17, 21 and local maxima at x = 3, 7, 11, 15, 19

$\Rightarrow m_1 = 6, n_1 = 5$

$f_2(x) = 98(x-1)^{50} - 600(x-1)^{49} + 2450$

$f_2'(x) = 98 \times 50(x-1)^{49} - 600 \times 49 \times (x-1)^{48}$

= 98 × 50 × (x – 1)48 (x – 7)

$f_2(x)$ has local minimum at x = 7 and no local maximum.

$\Rightarrow m_2 = 1, n_2 = 0$

Question Stem for Question Nos. 11 and 12

Let $g_i : \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \to R, i = 1, 2$, and $f : \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \to R$ be functions such that $g_1(x) = 1$, $g_2(x) = |4x - \pi|$ and $f(x) = \sin^2 x$, for all $x \in \left[\frac{\pi}{8}, \frac{3\pi}{8}\right]$. Define $S_i = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} f(x).g_i(x)dx, \ i = 1, 2$

[:Q.11] The value of $\frac{16S_1}{\pi}$ is ________ .

[:ANS] **02.00**

[:SOLN]

$$S_1 = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \sin^2 x.1dx$$

$$= \frac{1}{2}\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} (1 - \cos 2x)dx$$

$$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)_{\frac{\pi}{8}}^{\frac{3\pi}{8}}$$

$$S_1 = \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{8}$$

$$\Rightarrow \frac{16S_1}{\pi} = 2$$

[:Q.12] The value of $\frac{48S_2}{\pi^2}$ is ________ .

[:ANS] 01.50

[:SOLN]

$$S_2 = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \sin^2 x.|4x - \pi| dx$$

$$= \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} 4\sin^2 x \left|x - \frac{\pi}{4}\right| dx$$

Let $x - \frac{\pi}{4} = t \Rightarrow \; dx = dt$

$$S_2 = \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} 4\sin^2\left(\frac{\pi}{4} + t\right)|t| dt$$

$$= \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} 2(1 - \cos 2\left(\frac{\pi}{4} + t)\right)|t| dt$$

$$= \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} (2 + 2\sin 2t)|t| dt$$

$$= 2\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} |t| dt + 2\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} |t| \sin(2t) dt$$

$$= 4\int_{0}^{\frac{\pi}{8}} t dt + 0$$

$$S_2 = 2t^2 \Big]_0^{\frac{\pi}{8}} = \frac{\pi^2}{32}$$

$$\Rightarrow \quad \frac{48 S_2}{\pi^2} = \frac{3}{2}$$

SECTION - 3 (Maximum Marks : 12)

- This section contains **TWO (02) paragraphs**. Based on each paragraph, there are **TWO (02)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme: Full Marks : +3 If ONLY the correct option is chosen;

  Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);

  Negative Marks : −1 In all other cases.

Paragraph Type Questions 13 to 14

Let $M = \{(x, y) \in R \times R : x^2 + y^2 \le r^2\}$, where r > 0. Consider the geometric progression $a_n = \frac{1}{2^{n-1}}, n = 1, 2, 3, \ldots$. Let $S_0 = 0$ and, for $n \geq 1$, let S_n denote the sum of the first n terms of this progression. For $n \geq 1$, let C_n denote the circle with center $(S_{n-1}, 0)$ and radius a_n, and D_n denote the circle with center (S_{n-1}, S_{n-1}) and radius a_n.

[:Q.13] Consider M with $r = \frac{1025}{513}$. Let k be the number of all those circles C_n that are inside M. Let ℓ be the maximum possible number of circles among these k circles such that no two circles intersect. Then

[:A] $k + 2l = 22$

[:B] $2k + l = 26$

[:C] $2k + 3l = 34$

[:D] $3k + 2l = 40$

[:ANS] D

[:SOLN]

$\because \quad a_n = \frac{1}{2^{n-1}}$ and $S_n = 2\left(1-\frac{1}{2^n}\right)$

For circles C_n to be inside M.

$$S_{n-1} + a_n < \frac{1025}{513}$$

$$\Rightarrow \quad S_n < \frac{1025}{513}$$

$$\Rightarrow \quad 1-\frac{1}{2^n} < \frac{1025}{1026} = 1-\frac{1}{1026}$$

$$\Rightarrow \quad 2^n < 1026$$

$$\Rightarrow \quad n \le 10$$

$\therefore$ Number of circles inside be 10 = k

Clearly alternate circles do not intersect each other i.e., C_1, C_3, C_5, C_7, C_9 do not intersect each other as wellas C_2, C_4, C_6, C_8 and C_{10} do not intersect each other hence maximum 5 set of circles do not intersect eachother.

$\therefore l = 5$

$\therefore 3K + 2l = 40$

$\therefore$ Option D is correct.

[:Q.14] Consider M with $r = \frac{(2^{199}-1)\sqrt{2}}{2^{198}}$. The number of all those circles D_n that are inside M is

[:A] 198

[:B] 199

[:C] 200

[:D] 201

[:ANS] B

[:SOLN]

$$\because \quad r = \frac{(2^{199}-1)\sqrt{2}}{2^{198}}$$

Now, $\sqrt{2}\,S_{n-1}+a_n<\left(\frac{2^{199}-1}{2^{198}}\right)\sqrt{2}$

$$2\sqrt{2}\left(1-\frac{1}{2^{n-1}}\right)+\frac{1}{2^{n-1}}<\left(\frac{2^{199}-1}{2^{198}}\right)$$

$$\therefore\quad 2\sqrt{2}-\frac{\sqrt{2}}{2^{n-2}}+\frac{1}{2^{n-1}}<2\sqrt{2}-\frac{\sqrt{2}}{2^{198}}$$

$$\frac{1}{2^{n-2}}\left(\frac{1}{2}-\sqrt{2}\right)<-\frac{\sqrt{2}}{2^{198}}$$

$$\frac{2\sqrt{2}-1}{2\cdot 2^{n-2}}>\frac{\sqrt{2}}{2^{198}}$$

$$2^{n-2}<\left(2-\frac{1}{\sqrt{2}}\right)2^{197}$$

$\therefore n\leq 199$

$\therefore$ Number of circles = 199

Option (B) is correct.

Paragraph Type Questions 15 to 16

Let $\psi_1:[0,\infty)\to\mathbb{R},\psi_2:[0,\infty)\to\mathbb{R},f:[0,\infty)\to\mathbb{R}$ and $g:[0,\infty)\to\mathbb{R}$ be functions such that f(0) = g(0) = 0,

$\psi_1(x)=e^{-x}+x, x\geq 0,$

$\psi_2(x)=x^2-2x-2e^{-x}+2, x\geq 0,$

$f(x)=\int_{-x}^{x}\left(|t|-t^2\right)e^{-t^2}\,dt, x>0$

And $g(x)=\int_0^{x^2}\sqrt{t}\,e^{-t}\,dt,\ x>0.$

[:Q.15] Which of the following statements is TRUE ?

[:A] $f\left(\sqrt{\ln 3}\right)+g\left(\sqrt{\ln 3}\right)=\frac{1}{3}$

[:B] For every x > 1, there exists an $\alpha\in(1,x)$ such that $\Psi_1(x)=1+\alpha x$

[:C] For every x >0, there exists a $\beta\in(0,x)$ such that $\Psi_2(x)=2x\left(\psi_1(\beta)-1\right)$

[:D] f is an increasing function on the interval $\left[0, \frac{3}{2}\right]$

[:ANS] C
[:SOLN]

$\because \quad g(x) = \int_0^{x^2} \sqrt{t}\, e^{-t}\, dt, \ x > 0$

Let $t = u^2 \Rightarrow dt = 2u\, du$

$\therefore \quad g(x) = \int_0^{x} u\, e^{-u^2} \cdot 2u\, du$

$= 2\int_0^{x} t^2\, e^{-t^2} dt \qquad \ldots(i)$

and $f(x) = \int_{-x}^{x} \left(|t| - t^2\right) e^{-t^2} dt, \ x > 0$

$\therefore \quad f(x) = 2\int_0^{x} \left(t - t^2\right) e^{-t^2} dt \qquad \ldots(ii)$

From equation (i) + (ii) : $f(x) + g(x) = \int_0^{x} 2te^{-t^2}\, dt$

Let $t^2 = P \Rightarrow 2t\, dt = dP$

$\therefore \quad f(x) + g(x) = \int_0^{x^2} e^{-P} dP = \left[-e^{-P}\right]_0^{x^2}$

$\therefore \quad f(x) + g(x) = 1 - e^{-x^2} \qquad \ldots(iii)$

$\therefore \quad f\left(\sqrt{ln\, 3}\right) + g\left(\sqrt{ln\, 3}\right) = 1 - e^{-ln 3} = 1 - \frac{1}{3} = \frac{2}{3}$

$\therefore$ Option (A) is incorrect.

From equation (ii) : $f'(x) = 2\left(x - x^2\right)e^{-x^2} = 2x(1-x)e^{-x^2}$

$\because$ $f(x)$ is increasing in (0, 1)

$\therefore$ Option (D) is incorrect

$\because \quad \Psi_1(x) = e^{-x} + x$

$\Rightarrow \quad \Psi'_1(x) = 1 - e^{-x} < 1$ for x > 1

Then for $\alpha \in (1, x)$, $\Psi_1(x) = 1 + \alpha x$ does not true for $\alpha > 1$.

$\therefore$ Option (B) is incorrect

Now $\psi_2(x) = x^2 - 2x - 2e^{-x} + 2$

$\Rightarrow \quad \psi'_2(x) = 2x - 2 + 2e^{-x}$

$\therefore \quad \psi'_2(x) = 2\psi_1(x) - 2$

From LMVT

$$\frac{\psi_2(x) - \psi_2(0)}{x - 0} = \psi'_2(\beta) \text{ for } \beta \in (\infty, x)$$

$\Rightarrow \quad \psi_2(x) = 2x(\psi_1(\beta) - 1)$

$\therefore$ Option (C) is correct.

[:Q.16] Which of the following statements is TRUE ?

[:A] $\Psi_1(x) \le 1$, for all x > 0

[:B] $\Psi_2(x) \le 0$, for all x > 0

[:C] $f(x) \ge 1 - e^{-x^2} - \frac{2}{3}x^3 + \frac{2}{5}x^5$, for all $x \in \left(0, \frac{1}{2}\right)$

[:D] $g(x) \le \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7$, for all $x \in \left(0, \frac{1}{2}\right)$

[:ANS] **D**

[:SOLN]

$\because \quad \Psi_1(x) = e^{-x} + x$

and for all x > 0, $\Psi_1(x) > 1$

$\therefore$ (A) is not correct

$\Psi_1(x) = x^2 + 2 - 2(e^{-x} + x) > 0$ for $x > 0$

$\therefore$ (B) is not correct

Now, $\sqrt{t}\, e^{-t} = \sqrt{t}\left(1-\frac{t}{1!}+\frac{t^2}{2!}-\frac{t^3}{3!}+\ldots\ldots\ldots\infty\right)$

and $\sqrt{t}\, e^{-t} \le t^{\frac{1}{2}} - t^{\frac{3}{2}} + \frac{1}{2}t^{\frac{5}{2}}$

$$\therefore \quad \int_0^{x^2} \sqrt{t}\, e^{-t}\, dt \le \int_0^{x^2}\left(t^{\frac{1}{2}} - t^{\frac{3}{2}} + \frac{1}{2}t^{\frac{5}{2}}\right) dt$$

$$= \frac{2}{3}x^3 - \frac{2}{3}x^5 + \frac{1}{7} + \frac{1}{7}x^7$$

$\therefore$ Option (D) is correct.

and $f(x) = \int_{-x}^{x} (|t| - t^2)e^{-t^2} dt$

$$= 2\int_0^x (t - t^2)e^{-t^2} dt$$

$$= \int_0^x 2te^{-t^2} dt - 2\int_0^x t^2 e^{-t^2} dt$$

$$= 1 - e^{-x^2} - 2\int_0^x t^2 e^{-t^2} dt$$

$$\therefore \quad f(x) \le 1 - e^{-x^2} - 2\int_0^x t^2\left(1 - t^2\right) dt$$

$$= 1 - e^{-x^2} - 2\frac{x^3}{3} + \frac{2}{5} \text{ for all } x\left(0, \frac{1}{2}\right)$$

$\therefore$ Option (C) is correct.

SECTION - 4 (Maximum Marks : 12)

- This section contains **THREE (03)** questions.
- The answer to each question is a **NON-NEGATIVE INTEGER.**
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme: Full Marks : +4 If ONLY the correct integer is entered;

  Zero Marks : 0 In all other cases.

[:Q.17] A number is chosen at random from the set {1,2,3,…,2000}. Let p be the probability that the chosen number is a multiple of 3 or a multiple of 7. Then the value of $500p$ is _________ .

[:ANS] **214**

[:SOLN] E = a number which is multiple of 3 or multiple of 7

n(E) = (3, 6, 9, ..., 1998) + (7, 14, 21, ..., 1995) – (21, 42, 63, ... 1995)

n(E) = 666+ 285 – 95

n(E) = 856

n(E) = 2000

$$P(E) = \frac{856}{2000}$$

$$P(E) \times 500 = \frac{856}{4} = 214.$$

[:Q.18] Let E be the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. For any three distinct points P,Q and Q' on E, let $M(P,Q)$ be the mid-point of the line segment joining P and Q, and $M(P,Q')$ be the mid-point of the line segment joining P and Q'. Then the maximum possible value of the distance between $(P,)$ and $(P,')$, as P,Q and Q' vary on E, is ___ .

[:ANS] **4**

[:SOLN]

Let P(α),Q(β), Q'(γ)

$$M = \left(\frac{1}{2}(4\cos\alpha + 4\cos\theta), \frac{1}{2}(3\sin\alpha + 3\sin\beta)\right)$$

$$M' = \left(\frac{1}{2}(4\cos\alpha + 4\cos\gamma), \frac{1}{2}(3\sin\alpha + 3\sin\gamma)\right)$$

$$MM' = \frac{1}{2}\sqrt{(4\cos\alpha - 4\cos\beta)^2 + (3\sin\alpha - 3\sin\beta)^2}$$

$MM' = \frac{1}{2}$ distance between Q and Q′

MM′ is not depending on P

Maximum of QQ′ is possible when QQ′ = major axis

$$QQ' = 2(4) = 8$$

$$MM' = \frac{1}{2}.(QQ')$$

$$MM' = 4.$$

[:Q.19] For any real number x, let $[x]$ denote the largest integer less than or equal to x. If $I = \int_0^{10}\left[\sqrt{\frac{10x}{x+1}}\right]dx$, then the value of $9I$ is ___________.

[:ANS] **182.00**

[:SOLN]

$$I = \int_0^{10}\left[\sqrt{\frac{10x}{x+1}}\right]dx$$

$$y = \frac{10x}{x+1} \quad , \quad 0 \le x \le 10$$

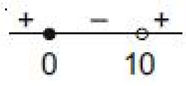
$$xy + y = 10x$$

$$x = \frac{y}{10-y}$$

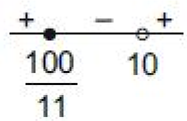
$$0 \le \frac{y}{10-y} \le 10$$

$$\frac{y}{10-y} \ge 0 \quad \text{and} \quad \frac{y}{10-y} - 10 \le 0$$

$$\frac{y}{y-10} \le 0 \quad \text{and} \quad \frac{11y-100}{y-10} \ge 0$$



and



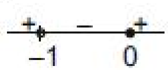
$y \in [0, 10)$ and $y \in \left(-\infty, \frac{100}{11}\right] \cup (10, \infty)$

$y \in \left[0, \frac{100}{11}\right]$

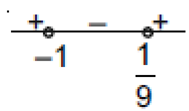
$\sqrt{y} \in \left[0, \frac{10}{\sqrt{11}}\right]$ $\Rightarrow$ $\left[\sqrt{y}\right] = \{0, 1, 2, 3\}$

Cose I : $0 \le \frac{10x}{x+1} < 1$

$\frac{10x}{x+1} \ge 0$ and $\frac{10x}{x+1} - 1 < 0$



and $\frac{9x-1}{x+1} < 0$



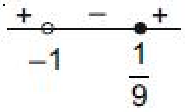
$x \in (-\infty, -1) \cup [0, \infty)$ and $x \in \left(-1, \frac{1}{9}\right)$

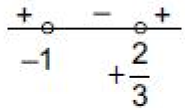
$x \in \left[0, \frac{1}{9}\right)$ then $\left[\sqrt{\frac{10x}{x+1}}\right] = 0$

Case II : $1 \le \frac{10x}{x+1} < 4$

$\frac{10x}{x+1} - 1 \ge 0$ and $\frac{10x}{x+1} - 4 < 0$

$\frac{9x-1}{x+1} \ge 0$ and $\frac{6x-4}{x+1} < 0$



and


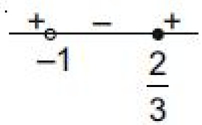
$x \in (-\infty, -1) \cup \left[\frac{1}{9}, \infty\right)$ and $x \in \left(-1, \frac{2}{3}\right)$

$x \in \left[\frac{1}{9}, \frac{2}{3}\right)$, $\left[\sqrt{\frac{10x}{x+1}}\right] = 1$

Case III : $4 \le \frac{10x}{x+1} < 9$

$\frac{10x}{x+1} - 4 \ge 0$ and $\frac{10x}{x+1} < 9$

$\frac{6x-4}{x+1} \ge 0$ and $\frac{x-9}{x+1} < 0$



and


$x \in (-\infty, -1) \cup \left[\frac{2}{3}, \infty\right)$ $x \in (-1, 9)$

$x \in \left[\frac{2}{3}, 9\right)$; $\left[\sqrt{\frac{10x}{x+1}}\right] = 2$

Case IV : $x \in [9, 0] \Rightarrow \left[\sqrt{\frac{10x}{x+1}}\right] = 3$

$$I = \int_0^{\frac{1}{9}} 0 \cdot dx + \int_{\frac{1}{9}}^{\frac{2}{3}} 1 \cdot dx + \int_{\frac{2}{3}}^{9} 2 \cdot dx + \int_9^{10} 3 \cdot dx$$

$$I = \left(\frac{2}{3} - \frac{1}{9}\right) + 2\left(9 - \frac{2}{3}\right) + 3(10-9)$$

$$I = \frac{5}{9} + \frac{50}{3} + 3$$

$9I = 182.$